

Remark on Algorithm 723: Fresnel Integrals

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“Algorithm 723: Fresnel Integrals” has been improved to provide more precise results for $x \gg 0$.

Categories and Subject Descriptors: D.3.2 [**Programming Languages**]: Language Classifications—*Fortran*; G.1.2 [**Numerical Analysis**]: Approximation—*rational approximation*; G.4 [**Mathematics of Computing**]: Mathematical Software—*certification and testing*

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Concerning evaluation of Fresnel Cosine and Sine integrals $C(x)$ and $S(x)$ for $x > 1.6$, the last sentence in the description of Algorithm 723 [Snyder 1993] states “Extensive accuracy testing would simply have validated the trigonometric function routines.”

Alfred H. Morris, Jr. has correctly pointed out that the procedures described in Snyder [1993] to compute $C(x)$ and $S(x)$ for $x > 1.6$ require computing $\sin \pi x^2/2$ and $\cos \pi x^2/2$. Thus the quoted statement begs the question whether one might compute these trigonometric functions more accurately than can be done by naive application of the Fortran intrinsic trigonometric function routines, when x is large.

The answer is in the affirmative. By separating x into its integer and fractional parts one can compute these functions with substantially better accuracy.

Let $x = n + f$ where $n = \lfloor x \rfloor$, and let $n = 2k + j$, where j is $n \bmod 2$. Then $\sin \pi x^2/2 = \sin \pi(4k^2 + 4kj + j^2 + 4kf + 2jf + f^2)/2$. By application of simple trigonometric identities, one observes that this is equal to $\sin \pi(2kf + f^2/2)$ or $\cos \pi(2kf + f + f^2/2)$, depending on whether j is zero or one, respectively. Letting $kf = m + g$, where $m = \lfloor kf \rfloor$, $\sin \pi x^2/2$ becomes $\sin \pi(2g + f^2/2)$ or $\cos \pi(2g + f + f^2/2)$, depending on whether j is zero or

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Table I. Summary of Errors Committed by $C(x)$ in Algorithm 723, as Published

Range	b:	Percent of Samples with b Bits Incorrect							
		1/2	1	2	3	4	5	6	7
$0 \leq x \leq 6$		39.0	25.9	21.0	10.7	3.2	0.3	0.0	—
$6 \leq x \leq 11$		18.7	15.2	20.5	22.7	16.7	5.8	0.3	0.0
$12 \leq x \leq 18$		13.1	10.4	15.6	21.2	22.6	14.0	3.1	0.0

 Table II. Summary of Errors Committed by $S(x)$ in Algorithm 723, as Published

Range	b:	Percent of Samples with b Bits Incorrect							
		1/2	1	2	3	4	5	6	7
$0 \leq x \leq 6$		25.7	25.2	24.0	11.5	3.1	0.2	0.0	—
$6 \leq x \leq 12$		18.8	14.8	20.1	23.6	16.6	5.8	0.4	0.0
$12 \leq x \leq 18$		12.7	10.8	15.6	21.9	22.8	13.2	3.0	0.0

one, respectively. A similar development may be used to improve computation of $\cos \pi x^2/2$.

In the procedures of Algorithm 723, we have replaced usage of Fortran intrinsic trigonometric function routines to calculate $\sin \pi x^2/2$ and $\cos \pi x^2/2$ by usage of routines that implement these ideas. A revised algorithm has been submitted to the Collected Algorithms from ACM.

TESTING

We tested the accuracy of the single-precision procedures of Algorithm 723 and the procedures of the revised algorithm, in computing $C(10^x)$ and $S(10^x)$, by subdividing each of the ranges $0 \leq x \leq 6$, $6 \leq x \leq 12$, and $12 \leq x \leq 18$ into 20,000 equal intervals, choosing a value of x from a uniform pseudorandom distribution over each interval and comparing the result to one computed by the double-precision procedures of Algorithm 723.

Test values were calculated using IEEE format 32-bit floating-point numbers; reference values were calculated using IEEE format 64-bit floating-point numbers.

Let A be absolute error, U the error measured in ULP, E the exponent of the floating-point result, f the fraction of the floating-point result (assumed to be normalized), r the radix of floating-point numbers, and ν the number of base- r digits in f . Then $U = A/r^{E-\nu}$. The more familiar relative error R (in multiples of r^ν) = U/f . Since in a normalized base- r floating-point number, $1/r \leq f < 1$, $U \leq R < rU$.

In Tables I–III the heading *b Bits Incorrect* indicates the number of low-order bits of the result that were incorrectly computed. $b = \lceil \log_2 U \rceil$.

Clearly, the revised procedures provide a substantial improvement.

The modifications described here allow us to change the formula $|x| > 2.0/\rho^{1/2}$ in the last paragraph on page 453 of Snyder [1993] to $|x| > 2.0/\rho$.

Table III. Summary of Errors Committed by Revised Algorithm 723

Range	b:	Percent of Samples of $C(x)$ with b Bits Incorrect					Percent of Samples of $S(x)$ with b Bits Incorrect				
		1/2	1	2	3	4	1/2	1	2	3	4
$0 \leq x \leq 6$		63.2	28.7	7.4	0.6	0.0	60.4	29.1	9.6	0.9	0.0
$6 \leq x \leq 11$		73.8	25.5	0.7	0.0	—	73.7	25.6	0.7	0.0	—
$12 \leq x \leq 18$		74.3	25.5	0.2	0.0	—	74.8	25.0	0.3	0.0	—

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