# The Gravitational Field of a Disk *

Fred T. Krogh, Edward W. Ng, William V. Snyder
*Jet Propulsion Laboratory, Pasadena, Calif., U. S. A.*

**Celestial Mechanics 26**, 4 (April 1982) pp 395-405.
Received March, 1979; accepted June 1981.
17 February 2013 corrections incorporated 2 July 2022.

**Abstract**

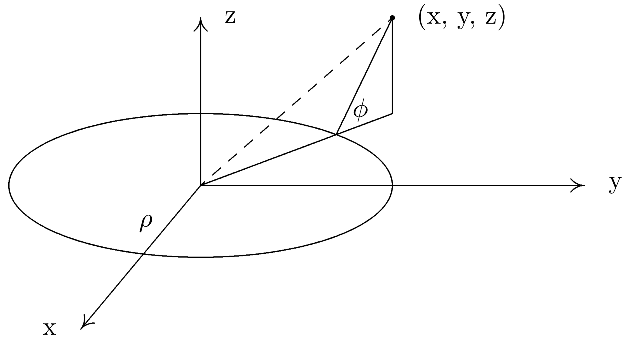
This note gives the gravitational potential of the disk $\{(x, y, z) : x^2 + y^2 \leq \rho^2, z = 0\}$ and the gravitational field at the point $(x, y, z)$. Formulas for a ring can be obtained as the difference of our results for two different values of $\rho$. Results are obtained in terms of elliptic integrals and we indicate how these functions can be computed efficiently. Formulas necessary for the computation of partial derivatives are also given.

## 1 Introduction

In a number of gravitational and electromagnetic problems, there is a need to calculate the field or potential due to a uniform disk or ring. For example, at the Jet Propulsion Laboratory, this need recently arose in connection with the study of Saturn's rings. The literature is varied in notation, accuracy and detail. It is therefore useful to give the expressions for both the potential and the field, and suggest efficient methods of computation. This section gives the expressions; Section 2 gives the derivation for the field; Section 3 provides computational methods; and the Appendix contains special asymptotic cases.

Consider the following disk with radius $\rho$.

*This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract NAS7-100, sponsored by the National Aeronautics and Space Administration.




For convenience these notations are used:

$$r^2 = x^2 + y^2, \tag{1}$$

$$R^2 = r^2 + z^2, \tag{2}$$

$$\delta^2(\theta) = R^2 + \rho^2 + 2r\rho\cos\theta \tag{3}$$

$$k^2 = 4r\rho/(R^2 + \rho^2 + 2r\rho) = 4r\rho/\delta^2(0), \tag{4}$$

$$\alpha^2 = 4r\rho/(\rho + r)^2, \tag{5}$$

$$\sin(\phi) = z/(R^2 + \rho^2 - 2r\rho)^{1/2} = z/\delta(\pi), \tag{6}$$

$K(k) =$ complete elliptic integral of the 1st kind,

$E(k) =$ complete elliptic integral of the 2nd kind,

$\Pi(\alpha^2, k) =$ complete elliptic integral of the 3rd kind,

$\Lambda_0(\phi, k) =$ Heuman's lambda function,

$$c = G\sigma, \tag{7}$$

$G =$ gravitational constant,

$\sigma =$ density of disk.

The potential is given by

$$V = 2c\left[z\left\{\frac{\pi}{2} + \frac{\pi}{2}\,\mathrm{sign}(\rho - r)\right\} - \frac{2\sqrt{r\rho}}{k}E(k) - \frac{k(\rho^2 - r^2)}{2\sqrt{r\rho}}K(k) - \right. \\ \left. - \frac{\rho - r}{\rho + r}\frac{kz^2}{2\sqrt{r\rho}}\Pi(\alpha^2, k)\right] \tag{8}$$

The attractions are given in cartesian coordinates:

$$x'' = -\frac{4cx\rho^{1/2}}{kr^{3/2}}\left[\left(1 - \frac{k^2}{2}\right)K(k) - E(k)\right], \tag{9}$$

$$y'' = -\frac{4cy\rho^{1/2}}{kr^{3/2}}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right], \tag{10}$$

$$z'' = \frac{czk}{\sqrt{r\rho}}K(k) - c\pi\,\mathrm{sign}(z)\left\{1 + \mathrm{sign}(\rho - r)\left[1 - \Lambda_0(\phi, k)\right]\right\} \tag{11}$$

The potential function has been derived by Blitzer and Lass [1] via an alternative approach. We have independently checked their result against ours via differentiation. It is listed here for completeness.

## 2 Derivations for the Attractions

By Newton's law of gravitation, the acceleration vector at the point $(x, y, z)$ due to an infinitesimal element of the disk at $(\xi, \eta)$ may be expressed readily in differential notations. We shall develop convenient formulas for the three cartesian components $(x'', y'', z'')$ of the acceleration. First consider the $x$-component:

$$x'' = -\frac{c}{\Delta^2}\frac{x-\xi}{\Delta}\,\mathrm{d}\xi\,\mathrm{d}\eta, \tag{12}$$

where $\Delta = \left[(x-\xi)^2 + (y-\eta)^2 + z^2\right]^{1/2}$. For the case of uniform density, integrating over the disk,

$$x'' = -c\int_{-\rho}^{\rho}\mathrm{d}\eta\int_{-\sqrt{\rho^2-\eta^2}}^{\sqrt{\rho^2-\eta^2}}\frac{(x-\xi)}{\Delta^3}\,\mathrm{d}\xi\,. \tag{13}$$

Clearly

$$x'' = -c\int_{-\rho}^{\rho}\left.\frac{1}{\Delta}\right|_{\xi=-\sqrt{\rho^2-\eta^2}}^{\sqrt{\rho^2-\eta^2}}\mathrm{d}\eta. \tag{14}$$

With the substitution $\eta = \rho\sin\theta$, this becomes

$$x'' = -c\rho\int_{-\pi/2}^{\pi/2}\left[\frac{\cos\theta}{\left[(\rho\cos\theta - x)^2 + (\rho\sin\theta - y)^2 + z^2\right]^{1/2}} - \frac{\cos\theta}{\left[(\rho\cos\theta + x)^2 + (\rho\sin\theta - y)^2 + z^2\right]^{1/2}}\right]\mathrm{d}\theta. \tag{15}$$

Let $\sin\psi = y/r$, $\cos\psi = x/r$. Then Equation (15) can be written

$$x'' = -c\rho\int_{-\pi/2}^{\pi/2}\left[\frac{\cos\theta}{\left[R^2 + \rho^2 - 2r\rho\cos(\theta - \psi)\right]^{1/2}} - \frac{\cos\theta}{\left[R^2 + \rho^2 + 2r\rho\cos(\theta + \psi)\right]^{1/2}}\right]\mathrm{d}\theta. \tag{16}$$

Replacing $\theta$ with $\psi - \theta + \pi$ in the first term of the integrand and with $\theta - \psi$ in the second term, there results

$$\begin{aligned} x'' &= c\rho \int_{\psi+\pi/2}^{\psi+3(\pi/2)} \frac{\cos(\theta-\psi)}{\delta(\theta)}\,\mathrm{d}\theta + c\rho \int_{\psi-\pi/2}^{\psi+\pi/2} \frac{\cos(\theta-\psi)}{\delta(\theta)}\,\mathrm{d}\theta \\ &= c\rho \int_{\psi-\pi/2}^{\psi+3(\pi/2)} \frac{\cos(\theta-\psi)}{\delta(\theta)}\,\mathrm{d}\theta. \end{aligned} \tag{17}$$

Since we are integrating over a full period, the integration limits can be anything that covers a full period. Thus

$$x'' = c\rho \int_{-\pi}^{\pi} \frac{\cos\theta\cos\psi + \sin\theta\sin\psi}{\delta(\theta)}\,\mathrm{d}\theta. \tag{18}$$

The first term in the numerator is an even function, and the second is odd (and hence integrates to zero). Thus

$$\begin{aligned} x'' &= \frac{2c\rho x}{r} \int_0^{\pi} \frac{\cos\theta}{\delta(\theta)}\,\mathrm{d}\theta = \frac{4c\rho x}{r} \int_0^{\pi/2} \frac{\cos 2\theta}{\delta(2\theta)}\,\mathrm{d}\theta \\ &= \frac{4c\rho x}{r} \int_0^{\pi/2} \frac{(1-2\sin^2\theta)}{\left[R^2+\rho^2+2r\rho-4r\rho\sin^2\theta\right]^{1/2}}\,\mathrm{d}\theta. \end{aligned} \tag{19}$$

$$\begin{aligned} x'' = \frac{2cxk\sqrt{\rho}}{r^{3/2}} \Bigg\{ & \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\left[1-k^2\sin^2\theta\right]^{1/2}} + \\ & + \frac{2}{k^2}\left[\int_0^{\pi/2} \frac{(1-k^2\sin^2\theta)}{\left[1-k^2\sin^2\theta\right]^{1/2}}\,\mathrm{d}\theta - \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\left[1-k^2\sin^2\theta\right]^{1/2}}\right]\Bigg\} \end{aligned} \tag{20}$$

where $k$ is defined in Equation (4). From the definition for elliptic integrals, see Equations (110.02) and (110.03) of [2], for example, Equation (20) can be written

$$\begin{aligned} x'' &= -\frac{4cx\sqrt{\rho}}{kr^{3/2}}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right] \\ &= -\frac{2cx\delta(0)}{r^2}\left[\left(1-\frac{k^2}{2}\right)K(k) - E(k)\right]. \end{aligned} \tag{21}$$

Note that although there is a great deal of cancellation in the computation of $x''$, and hence poor relative accuracy when $x''$ is small, this should not be of concern in a numerical integration where overall absolute error determines the precision of the final results.

The case of $y''$ is obtained simply by replacing $x$ with $y$ in Equation (21).

For the case of $z''$ we express the acceleration in polar coordinates, $\xi = P\cos\theta$, $\eta = P\sin\theta$.

$$z'' = -cz \int_{-\pi}^{\pi}\int_0^{\rho} \frac{P}{[(x-P\cos\theta)^2 + (y-P\sin\theta)^2 + z^2]^{3/2}}\,\mathrm{d}P\,\mathrm{d}\theta. \tag{22}$$

The term to the 3/2's power in the denominator can be written as

$$R^2 + P^2 - 2P(x\cos\theta + y\sin\theta) = R^2 + P^2 - 2rP\cos(\theta+\psi), \tag{23}$$

where $\psi$ is defined as for the $x''$ case. Then since the integration is over one complete cycle, $\psi$ can be set to $\pi$, giving

$$\begin{aligned} z'' &= -\,cz\int_{-\pi}^{\pi}\int_0^{\rho}\frac{P\,\mathrm{d}P\,\mathrm{d}\theta}{[R^2+P^2+2rP\cos\theta]^{3/2}} \\ &= cz\int_{-\pi}^{\pi}\left[\frac{rP\cos\theta + R^2}{(R^2-r^2\cos^2\theta)(R^2+P^2+2rP\cos\theta)^{1/2}}\right]_{P=0}^{\rho}\mathrm{d}\theta \\ &= 2cz\int_0^{\pi}\left[\frac{r\rho\cos\theta+R^2}{(R^2-r^2\cos^2\theta)\delta(\theta)} - \frac{R}{R^2-r^2\cos^2\theta}\right]\mathrm{d}\theta. \end{aligned} \tag{24}$$

$$\begin{aligned} \int_0^{\pi}\frac{R}{R^2-r^2\cos^2\theta}\,\mathrm{d}\theta &= \frac{1}{2}\int_0^{\pi}\left[\frac{1}{R+r\cos\theta}+\frac{1}{R-r\cos\theta}\right]\mathrm{d}\theta = \\ &= \frac{1}{(R^2-r^2)^{1/2}}\left[\tan^{-1}\left[\frac{(R^2-r^2)^{1/2}\tan\frac{1}{2}\theta}{R+r}\right]+\right. \\ &\qquad \left.+\tan^{-1}\left[\frac{(R^2-r^2)^{1/2}\tan\frac{1}{2}\theta}{R-r}\right]\right]_{\theta=0}^{\pi} = \\ &= \pi/(R^2-r^2)^{1/2} = (\pi\,\mathrm{sign}\,z)/z. \end{aligned} \tag{25}$$

$$\begin{aligned} \int_0^{\pi}\frac{(r\rho\cos\theta+R^2)}{(R^2-r^2\cos^2\theta)\delta(\theta)}\,\mathrm{d}\theta &= 2\int_0^{\pi/2}\frac{(r\rho\cos 2\theta+R^2)}{(R^2-r^2\cos^2 2\theta)\delta(2\theta)}\,\mathrm{d}\theta \\ = 2\int_0^{\pi/2} & \frac{R^2+r\rho-2r\rho\sin^2\theta}{\left[R^2-r^2(1-2\sin^2\theta)^2\right](R^2+\rho^2+2r\rho-4r\rho\sin^2\theta)^{1/2}}\,\mathrm{d}\theta \\ = \int_0^{\pi/2} & \frac{1}{(R^2+\rho^2+2r\rho-4r\rho\sin^2\theta)^{1/2}}\left[\frac{R-\rho}{R+r(1-2\sin^2\theta)}+\right. \\ & \left.+\frac{R+\rho}{R-r(1-2\sin^2\theta)}\right]\mathrm{d}\theta \\ = \frac{k}{2\sqrt{r\rho}} & \int_0^{\pi/2}\frac{1}{(1-k^2\sin^2\theta)^{1/2}}\left[\frac{(R-\rho)/(R+r)}{\left(1-\frac{2r}{R+r}\sin^2\theta\right)}+\right. \\ & \left.+\frac{(R+\rho)/(R-r)}{1+\frac{2r}{R-r}\sin^2\theta}\right]\mathrm{d}\theta, \end{aligned} \tag{26}$$

where $k$ is defined by Equation (4). Equation (26) gives the sum of two elliptic integrals of the third kind, see for example Equation (110.04) of [2]. Thus

Equations (24)–(26) yield

$$
\begin{aligned}
z'' = & -2c\pi \operatorname{sign} z + \frac{czk}{\sqrt{r\rho}} \left[ \left( \frac{R-\rho}{R+r} \right) \Pi \left( \frac{2r}{R+r}, k \right) + \right. \\
& \left. + \left( \frac{R+\rho}{R-r} \right) \Pi \left( \frac{-2r}{R-r}, k \right) \right] \\
= & \, 2c \operatorname{sign} z \left\{ -\pi + \frac{k}{2\sqrt{r\rho}} \left[ (R-\rho) \left( \frac{R-r}{R+r} \right)^{1/2} \Pi \left( \frac{2r}{R+r}, k \right) + \right. \right. \\
& \left. \left. + \ (R+\rho) \left( \frac{R+r}{R-r} \right)^{1/2} \Pi \left( \frac{-2r}{R-r}, k \right) \right] \right\} .
\end{aligned}
\tag{27}
$$

Since elliptic integrals of the third kind are more difficult to compute than the other two, we go through some transformations to replace the two $\Pi$'s in Equation (27). Using Equations (410.01) and (413.01) from [2], one obtains

$$
z'' = \frac{2czk}{(R+\rho)} \sqrt{\frac{\rho}{r}} \, K(k) + \pi c \operatorname{sign} z \left[ -2 + \Lambda_0(\xi, k) + \Lambda_0(\psi, k) \right] , \tag{28}
$$

where $\Lambda_0$ is the Heuman lambda function and

$$
\begin{aligned}
\sin(\xi) &= \frac{R-\rho}{\delta(\pi)} \\
\sin(\psi) &= \frac{2\sqrt{r\rho}}{k(R+\rho)}
\end{aligned}
\tag{29}
$$

The addition formula (152.01) of [2] then gives

$$
z'' = \frac{czk}{\sqrt{r\rho}} \, K(k) + \pi c \operatorname{sign} z \left[ -2 + \Lambda_0(\phi, k) \right] \tag{30}
$$

where $\sin(\phi)$ is defined by Equation (6), and

$$
\cos(\phi) = \frac{\rho - r}{\delta(\pi)}, 0 \leq \phi \leq \pi. \tag{31}
$$

# 3 Computational Approaches

For actual computation we recommend two alternatives. One is to replace $\Lambda_0$ in Equation (30) with the identity (150.03) of [2] and using the fact that $\Lambda_0(\pi - \phi, k) = 2 - \Lambda_0(\phi, k)$, obtain

$$
\begin{aligned}
z'' = & \frac{czk}{\sqrt{r\rho}} \, K(k) - 2c \operatorname{sign} z \left\{ \frac{\pi}{2} + \frac{\pi}{2} \operatorname{sign}(\rho - r) - \right. \\
& \left. - \operatorname{sign}(\rho - r) \left[ (E(k) - K(k)) F(\theta, k') + K(k) E(\theta, k') \right] \right\} ,
\end{aligned}
\tag{32}
$$

where $F(\theta, k')$ and $E(\theta, k')$ are incomplete elliptic integrals of the first and second kind, $k' = (1 - k^2)^{1/2}$, and $\theta = \tan^{-1} |z/(r - \rho)|$. (Note that $\theta = \phi$ if $0 \leq \phi \leq \pi/2$ and $\theta = \pi - \phi$ for $\pi/2 \leq \phi \leq \pi$.) This alternative requires good algorithms or subroutines for the complete and incomplete integrals. For $K(k)$ and $E(k)$ Cody's approximations [3] and software [4] are recommended. For the incomplete integrals we refer to the various approaches recommended by Bulirsch in [5], [6], [7] and Carlson in [8], [9], [10].

A second very attractive alternative is to use the cel function defined by Bulirsch ([5], p. 307). All the relevant elliptic integrals in this paper may be expressed in terms of the cel function of Bulirsch's formulas:

$$k_c^2 = 1 - k^2;\ p = \alpha^2 + 1 \tag{33}$$

$$K(k) = \mathrm{cel}(k_c, 1, 1, 1) \tag{34}$$

$$E(k) = \mathrm{cel}(k_c, 1, 1, k_c^2) \tag{35}$$

$$\Pi(\alpha^2, k) = \mathrm{cel}(k_c, p, 1, 1) \tag{36}$$

$$\Lambda_0(\phi, k) = \frac{2}{\pi}\sqrt{q}\sin\phi\,\mathrm{cel}(k_c, q, 1, k_c^2), \tag{37}$$

where $q = 1 + k^2 \tan^2 \phi$.

A very efficient algorithm is provided in the above reference for computation of cel.

Partial derivatives of $x''$, $y''$ and $z''$ with respect to $x$, $y$ and $z$ can be computed from Equations (21), (30) and (31), the definitions or $r$ and $R$, and the formulas below which correspond with Equations (710.00), (710.02), (710.11) and (730.04) of [2].

$$\frac{\partial K(k)}{\partial k} = \frac{E(k) - (1 - k^2)K(k)}{k(1 - k^2)} \tag{38}$$

$$\frac{\partial E(k)}{\partial k} = \frac{1}{k}\left[E(k) - K(k)\right] \tag{39}$$

$$\frac{\partial \Lambda_0(\phi, k)}{\partial k} = \frac{2\left[E(k) - K(k)\right]\sin\phi\cos\phi}{\pi k\left[1 - (1 - k^2)\sin^2\phi\right]^{1/2}} \tag{40}$$

$$\frac{\partial \Lambda_0(\phi, k)}{\partial \phi} = \frac{2\left[E(k) - (1 - k^2)K(k)\sin^2\phi\right]}{\pi\left[1 - (1 - k^2)\sin^2\phi\right]^{1/2}} \tag{41}$$

Prof. B. C. Carlson has called our attention to [11] and [12] which give formulas for the $z''$ case. Both of these papers suffer from misprints (or perhaps errors in the case of [12]), and are so brief that one must almost rederive the results in order to be sure of using the formulas correctly.

# Appendix: Approximations for Special Cases

We give here approximations for $x''$ (and hence $y''$) and $z''$ for large $R$, and for $z''$ as $z \to 0$. These approximations are useful for estimating error in using a point mass as an approximation to the disk when far away, and also serve as a weak check on the results. For the case of $z''$ we also give simplified expressions for the cases $r = 0$ and $r = \rho$.

For $x''$, the expansions (900.00) and (900.07) of [2] substituted in Equation (21) give the following when $R$ is large.

$$\begin{aligned} x'' &= \left[\frac{-2cx}{r^2}\delta(0)\right]\left(\frac{\pi}{2}\right)\left\{(1-\frac{k^2}{2})\sum_{m=0}^{\infty}\frac{1^2\cdot 3^2\cdots(2m-1)^2}{4^m m!m!}k^{2m}+\right. \\ &\quad \left.+\sum_{m=0}^{\infty}\frac{1^2\cdot 3^2\cdots(2m-1)^2}{(2m-1)4^m m!m}k^{2m}\right\} \\ &= \frac{-\pi cx}{r^2}\delta(0)\sum_{m=2}^{\infty}\frac{2m(m-1)1^2\cdot 3^2\cdots(2m-1)^2}{(2m-1)^2 4^m m!m!}k^{2m} \\ &= \frac{-2\pi c\rho^2 x}{\delta^3(0)}\sum_{m=0}^{\infty}\frac{1^2\cdot 3^2\cdots(2m+1)^2}{m!(m+2)!}\left(\frac{k}{2}\right)^{2m}. \end{aligned} \tag{A1}$$

Note that this gives the usual result for a point mass in the limit as $\rho \to 0$. ($\pi c\rho^2$ is a constant times the mass of the disk.)

For the expansion of $z''$ for large $R$, we use Equation (30) and Equation (412.01) of [2] to obtain

$$z'' = -\pi c\,\mathrm{sign}\,z + \frac{czk}{(r+\rho)\sqrt{r\rho}}\left\{2\rho K(k) + (r-\rho)\Pi\left(\frac{k^2(r+\rho)^2}{4r\rho}, k\right)\right\}. \tag{A2}$$

Using expansions (900.00) and (906.05) from [2], with $\alpha^2 = k^2(r+\rho)^2/4r\rho$ we obtain

$$\begin{aligned} z'' &= \pi c\,\mathrm{sign}\,z\left\{-1+\frac{k(R^2-r^2)^{1/2}}{2\sqrt{r\rho}}\left[\frac{2\rho}{r+\rho}\sum_{m=0}^{\infty}\frac{1^2\cdot 3^2\cdots(2m-1)^2}{4^m m!m!}k^{2m}+\right.\right. \\ &\quad \left.\left.+\frac{r-\rho}{r+\rho}\sum_{m=0}^{\infty}\sum_{j=0}^{m}\frac{(2m)!(2j)!}{4^m 4^j m!m!j!j!}k^{2j}(a^2)^{m-j}\right]\right\} \\ &= \pi c\,\mathrm{sign}\,z\left\{-1+\frac{k(R^2-r^2)^{1/2}}{2\sqrt{r\rho}}\sum_{m=0}^{\infty}\left[\frac{1^2\cdot 3^2\cdots(2m-1)^2}{4^m m!m!}k^{2m}+\right.\right. \\ &\quad \left.\left.+\frac{r-\rho}{r+\rho}\sum_{j=0}^{m-1}\frac{(2m)!(2j)!}{4^m 4^j m!m!j!j!}k^{2j}(a^2)^{m-j}\right]\right\}. \end{aligned} \tag{A3}$$

Since

$$\left[\frac{R^2-r^2}{\delta^2(0)}\right]^{1/2} = \left[1-\frac{(r+\rho)^2}{\delta^2(0)}\right]^{1/2} = \left[1-\alpha^2\right]^{1/2} = \\ = 1-\sum_{m=1}^{\infty}\frac{1\cdot 3\cdots|2m-3|}{2^m m!}(\alpha^2)^m \tag{A4}$$

$$\begin{aligned} z'' = \pi c \operatorname{sign} z \Bigg\{ & \sum_{m=1}^{\infty}\Bigg[\frac{1^2\cdot 3^2\cdots(2m-1)^2}{4^m m! m!}k^{2m}+ \\ & + \frac{r-\rho}{r+\rho}\sum_{j=0}^{m-1}\frac{(2m)!(2j)!}{4^m 4^j m! m! j! j!}k^{2j}(a^2)^{m-j}\Bigg]- \\ & - \sum_{m=1}^{\infty}\frac{1\cdot 3\cdots|2m-3|}{2^m m!}\alpha^{2m}- \\ & - \sum_{m=1}^{\infty}\sum_{i=1}^{m}\left[\frac{1\cdot 3\cdots|2m-2i-1|}{2^{m-i+1}(m-i+1)!}(\alpha^2)^{m-i+1}\right]\times \\ & \times\left[\frac{1^2\cdot 3^2\cdots(2i-1)^2}{4^i i! i!}k^{2i}+\frac{r-\rho}{r+\rho}\sum_{j=0}^{i-1}\frac{(2i)!(2j)!}{4^i 4^j i! i! j! j!}k^{2j}(a^2)^{i-j}\right]\Bigg\}. \end{aligned} \tag{A5}$$

By rearranging terms this can be written

$$z'' = \pi c \operatorname{sign} z \sum_{m=1}^{\infty}\left[a_m k^{2m} - b_m\alpha^{2m} - \sum_{j=1}^{m-1}c_{mj}k^{2j}(\alpha^2)^{m-j}\right], \tag{A6}$$

where

$$\begin{aligned} a_m &= \frac{1^2\cdot 3^2\cdots(2m-1)^2}{4^m m! m!} \\ b_m &= \frac{1\cdot 3\cdots|2m-3|}{2^m m!}- \\ &\quad - \frac{r-\rho}{r+\rho}\left[\frac{(2m)!}{4^m m! m!}-\sum_{i=1}^{m-1}\frac{1\cdot 3\cdots|2m-2i-3|(2i)!}{2^{m+i}(m-i)! i! i!}\right] \\ c_{mj} &= \frac{1\cdot 3\cdots|2m-2j-3|\cdot 1^2\cdot 3^2\cdots(2j-1)^2}{2^{m+j}j! j!(m-j)!}- \\ &\quad - \frac{r-\rho}{r+\rho}\left[\frac{(2m)!(2j)!}{4^m 4^j(m! j!)^2}-\sum_{i=j+1}^{m-1}\frac{1\cdot 3\cdots|2m-2i-3|(2j)!(2i)!}{2^{m+i}4^j(m-i)!(i! j!)^2}\right]. \end{aligned}$$

Taking terms for $m = 1$ and $m = 2$ we get the approximation

$$\begin{aligned} z'' = \pi c \operatorname{sign} z \Bigg\{ \frac{k^2}{4} - \left[ \frac{1}{2} - \frac{r-\rho}{r+\rho} \left( \frac{1}{2} \right) \right] \alpha^2 + \frac{9}{64} k^4 - \\ - \left[ \frac{1}{8} - \frac{r-\rho}{r+\rho} \left( \frac{3}{8} - \frac{1}{4} \right) \right] \alpha^4 - \left[ \frac{1}{8} - \frac{r-\rho}{r+\rho} \frac{3}{16} \right] \alpha^2 k^2 \Bigg\} = \quad \text{(A7)} \\ = \; - \pi c \operatorname{sign} z \left\{ \frac{\rho^2}{\delta^2(0)} + \frac{(2r^2 - 8r\rho - \rho^2)\rho^2)}{4\delta^4(0)} \right\}. \end{aligned}$$

The above approximation is not useful for $z$ small. To get an approximation for this case, we use Equation (39) to obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} \Lambda_0(\sin^{-1} u, k) = \frac{2}{\pi} \frac{E(k) - (1-k^2)K(k)u^2}{\left[1 - (1-k^2)u^2\right]^{1/2} \left[1 - u^2\right]^{1/2}} \tag{A8}$$

Since

$$\begin{aligned} \left[1 - u^2\right]^{-1/2} \left[1 - (k'u)^2\right]^{-1/2} = \left[ 1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} u^{2m} \right] \times \\ \times \left[ 1 + \sum_{m=1}^{\infty} \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} (k'u)^{2m} \right] = \\ = 1 + \frac{1}{2}(1 + k'^2)u^2 + \sum_{m=2}^{\infty} \Bigg\{ \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} (1 + k'^2) + \\ + \sum_{i=1}^{m-1} \frac{1 \cdot 3 \cdots (2m-2i-1) \cdot 1 \cdot 3 \cdots (2i-1)}{2^m (m-i)! i!} k'^{2i} \Bigg\} u^{2m} \end{aligned}$$

we get from Equations (30), (31) and a Maclaurin's expansion with respect to $u$ of $\Lambda_0(\sin^{-1} u, k)$, where $u = \sin \phi$

$$\begin{aligned} z'' = \frac{czk}{\sqrt{r\rho}} K(k) + \pi c \operatorname{sign} z \Bigg\{ \quad -1 + \operatorname{sign}(r - \rho) \times \\ \times \Bigg[ 1 - 2c \operatorname{sign} z \int_0^{\sin\phi} \left\{ E(k) - (1-k^2)K(k)u^2 \right\} \Bigg\{ 1 + \frac{1}{2}(2-k^2)u^2 + \\ + \sum_{m=2}^{\infty} \Bigg[ \frac{1 \cdot 3 \cdots (2m-1)}{2^m m!} (2-k^2) + \\ + \sum_{i=1}^{m-1} \frac{1 \cdot 3 \cdots (2m-2i-1) \cdot 1 \cdot 3 \cdots (2i-1)}{2^m (m-1)! i!} (1-k^2)^i \Bigg] u^{2m} \Bigg\} \mathrm{d}u \Bigg] \Bigg\} \end{aligned} \tag{A9}$$

$$
\begin{aligned}
z'' = & -\pi c \operatorname{sign} z \, [1 - \operatorname{sign}(r-\rho)] + \frac{czk}{\sqrt{r\rho}} \, K(k) - \\
& - \frac{2cz \operatorname{sign}(r-\rho)}{\delta(\pi)} \left\{ E(k) + \frac{1}{3} \left[ \frac{1}{2}(2-k^2)E(k) - (1-k^2)K(k) \right] \times \right. \\
& \times \frac{z^2}{\delta^2(\pi)} + \frac{1}{5} \left[ (1 - \frac{5}{8}k^2)E(k) - \frac{1}{2}(1-k^2)(2-k^2)K(k) \right] \times \\
& \left. \times \left( \frac{z^2}{\delta^2(\pi)} \right)^2 + \cdots \right\} .
\end{aligned}
\tag{A10}
$$

For $r = 0$, we substitute in Equation (30) ($k = 0,\ K(0) = \pi/2,\ \Lambda_0(\phi, 0) = \sin\phi$) to get

$$
\begin{aligned}
z'' &= \frac{\pi c z}{(R^2+\rho^2)^{1/2}} + \pi c \operatorname{sign} z \left[ -2 - \frac{(R^2-r^2)^{1/2}}{(R^2+\rho^2)^{1/2}} \right] = \\
&= -2\pi c \left[ \operatorname{sign} z - \frac{z}{(z^2+\rho^2)^{1/2}} \right] .
\end{aligned}
\tag{A11}
$$

Finally for the case $r = \rho$, substitution into Equation (A2) gives

$$
z'' = -\pi c \operatorname{sign} z + 2czK(k)(R^2 + 3\rho^2)^{-1/2}. \tag{A12}
$$

## References


[1] Harry Lass and Leon Blitzer, *The gravitational potential due to uniform disks and rings*, **Celestial Mechanics 30**, 3 (1983) 225–228. Noted in the original publication as a Jet Propulsion Laboratory Inter-Office memorandum.

[2] Paul F. Byrd and Morris. D. Friedman, **Handbook of Elliptic Integrals for Engineers and Scientists**, Springer-Verlag, Berlin, Germany / Heidelberg, Germany / London, UK / etc., 2nd edition (1971).

[3] W. J. Cody, *Chebyshev approximations for the complete elliptic integrals $K$ and $E$*, **Mathematics of Computation 19**, 89–92 (1965) 105–112. See also [13].

[4] W. J. Cody, *The FUNPACK package of special function subroutines*, **ACM Transactions on Mathematical Software 1**, 1 (March 1975) 13–25.

[5] Roland Bulirsch, *Numerical calculation of elliptic integrals and elliptic functions*, **Numerische Mathematik 7** (1965) 78–90.

[6] Roland Bulirsch, *Numerical calculation of elliptic integrals and elliptic functions*, **Numerische Mathematik 7** (1965) 353–354.

[7] Roland Bulirsch, *Numerical calculation of elliptic integrals and elliptic functions*, **Numerische Mathematik 13** (1969) 305–315.

[8] Bille C. Carlson, *On computing elliptic integrals and functions*, **Journal of Mathematical Physics 44** (1965) 36–51.

[9] Bille C. Carlson, *Computing elliptic integrals by duplication*, **Numerische Mathematik 33** (1979) 1–16.

[10] Bille C. Carlson and Elaine M. Notis, *Algorithm 577 – Incomplete elliptic integrals*, **ACM Transactions on Mathematical Software 7**, 3 (1981).

[11] B. S. R. Rao and I. V. Radhakrishnamurthy, *An expression for the attraction of a circular plate*, **Indian J. Pure Appl. Phys. 4** (1966) 276–278.

[12] S. K. Singh, *Gravitational attraction of a circular disk*, **Geophys. 42** (1977) 111–113.

[13] W. J. Cody, Jr., *Corrigenda: "Chebyshev approximations for the complete elliptic integrals K and E"*, **Mathematics of Computation 20**, 93 (Jan. 1966) 207–207. See [3].